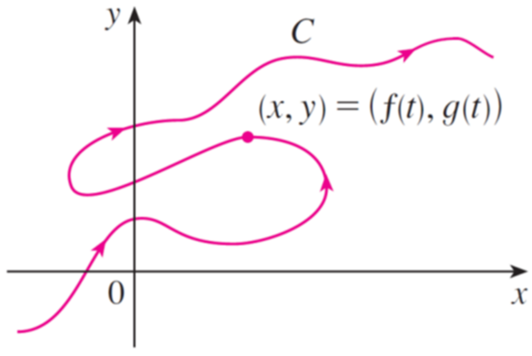
Curves Defined by Parametric Equations

Imagine that a particle moves along the curve C shown in the following figure. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test.



Suppose that x- and y-coordinates are both given as functions of a third variable t (called a parameter) in an interval I by the equations

$$x = f(t) \quad , \quad y = g(t), \quad \text{for } t \in I \quad \text{(called parametric equations on } I\text{)}$$

then

- each value of t determines a point (x, y), which we can plot in a coordinate plane,
- as t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve

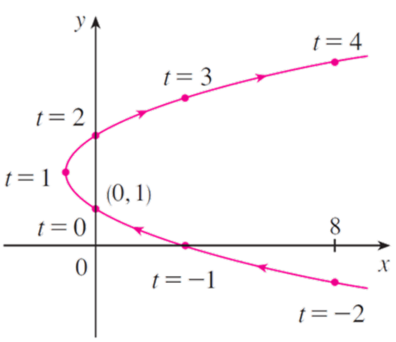
$$C = \{(x, y) \in \mathbb{R}^2 \mid x = f(t),\ y = g(t),\ t \in I\},$$

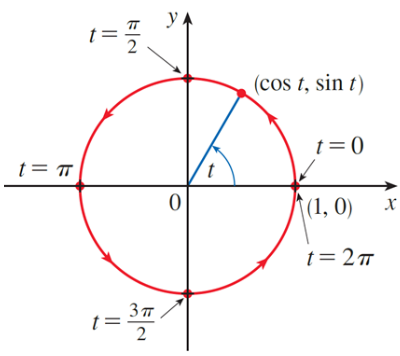
which we call a parametric curve on I.

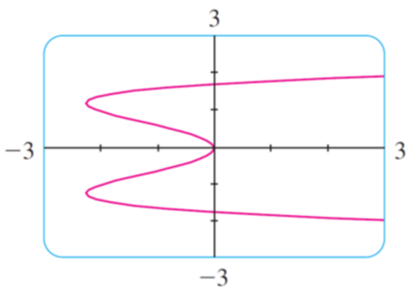
Example 1. Sketch and identify the curve defined by the parametric equations $x = t^2 - 2t,\ y = t + 1$ for $t \in \mathbb{R}$. [Solution: Since $t = y - 1$ and $x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$, the curve traced out by the particle is a parabola $x = (y-2)^2 - 1$.]

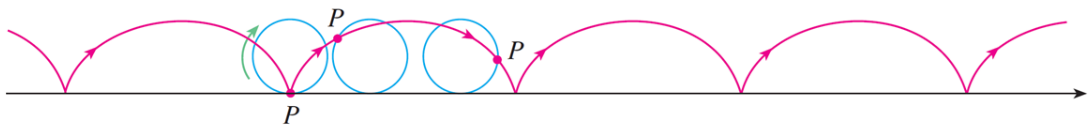
Example 2. Determine the curve represented by the following parametric equations $x = \cos t,\ y = \sin t$ for $0 \le t \le 2\pi$. [Solution: Since $x^2 + y^2 = 1$, the curve determined by the parametric equations is the unit circle $x^2 + y^2 = 1$.]

Example 3. Use a calculator or computer to graph the curve $x = y^4 - 3y^2$ [Hint: If $y = t$ then we can draw the curve determined by the parametric equations $x = t^4 - 3t^2,\ y = t$ for $t \in \mathbb{R}$.]





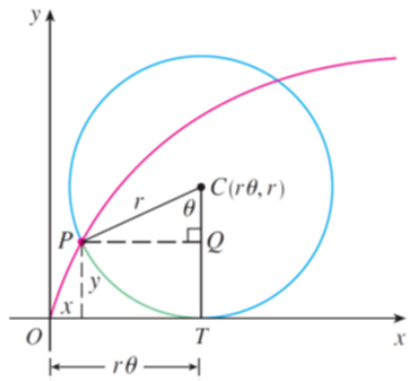




Example 4. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**.

If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians.



Because the circle has been in contact with the line, we see from the above figure that the distance it has rolled from the origin is

$$|OT| = \text{length of arc } \widehat{PT} = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from the above figure we see that

$$\begin{aligned} x = x(\theta) &= |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta) \\ y = y(\theta) &= |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta) \end{aligned}$$

Therefore parametric equations of the **cycloid** are

$$C: \quad x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta), \quad \theta \in \mathbb{R}$$

Calculus with Parametric Curves

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x.

Since the Chain Rule gives $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$, we obtain the following conclusions.

- If $\dfrac{dx}{dt} \neq 0$, then

(i) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$,

(ii) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{dt}{dx} = \dfrac{d}{dt}\left(\dfrac{dy/dt}{dx/dt}\right)\cdot\dfrac{1}{dx/dt} = \dfrac{d^2y/dt^2}{(dx/dt)^2} - \dfrac{(dy/dt)(d^2x/dt^2)}{(dx/dt)^3}$.

- The curve has a horizontal tangent at the point where $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$.
- The curve has a vertical tangent at the point where $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.
- Other methods needed to determine the slope of the tangent if both $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$.

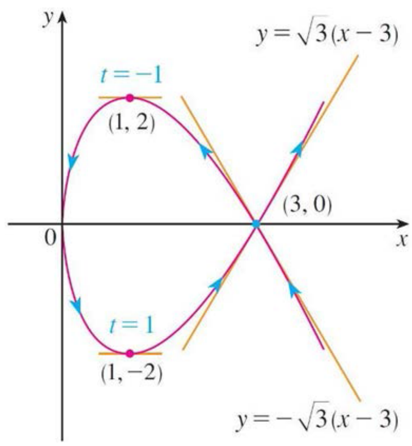
Example Let C be a curve defined by the parametric equations $x = t^2$, $y = t^3 - 3t$ for $t \in \mathbb{R}$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.
[Solution: Note that $x = 3$ for $t = \pm\sqrt{3}$ and, in both cases, $y = t(t^2 - 3) = 0$. Therefore the point $(3, 0)$ on C arises from two values of the parameter, $t = \pm\sqrt{3}$. This indicates that C crosses itself at $(3, 0)$.
Since $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 3}{2t}$, the slope of the tangents are $dy/dx = \pm\sqrt{3}$ when $t = \pm\sqrt{3}$, respectively. Thus we have two different tangent lines at $(3, 0)$ with equations $y = \pm\sqrt{3}(x - 3)$.]

(b) Find the points on C where the tangent is horizontal or vertical. [Solution: Since $dy/dt = 3t^2 - 3 = 0$ and $dx/dt = 2t \neq 0$ when $t = \pm 1$, C has a horizontal tangent at $(1, \mp 2)$. Since $dx/dt = 2t = 0$ and $dy/dt = 3t^2 - 3 \neq 0$ when $t = 0$, C has a vertical tangent at $(0, 0)$.]

(c) Determine where the curve is concave upward or downward. [Solution: Since $\dfrac{d^2y}{dx^2} = \dfrac{3t^2 + 3}{4t^3}$, C is concave upward when $t > 0$ and concave downward when $t < 0$.]

(d) Sketch the curve. Using the information from parts (b) and (c), we sketch C as follows



Areas

Recall that the area under a differentiable curve $y = F(x) \geq 0$ from a to b is given by

$$A = \int_a^b F(x)dx.$$

Suppose that

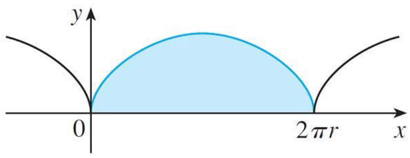
$$\begin{aligned} C &= \{(x,y) \mid y = F(x) \geq 0,\ x \in [a,b]\} \\ &= \{(x,y) \mid x = f(t),\ y = g(t) \geq 0 \text{ and } f : [\alpha,\beta] \to [a,b] \text{ is } 1-1 \text{ and onto}\} \end{aligned}$$

i.e. C is traced out once by the parametric equations $x = f(t)$ and $y = g(t) \geq 0$ for $t \in [\alpha,\beta]$

Then by using the substitution we can calculate the area as follows:

$$\begin{aligned} A = \int_a^b y\,dx &= \int_\alpha^\beta g(t)f'(t)\,dt \quad \text{if } f'(t) > 0,\ t \in [\alpha,\beta] \\ \text{or } &\int_\beta^\alpha g(t)f'(t)\,dt \quad \text{if } f'(t) < 0,\ t \in [\alpha,\beta]. \end{aligned}$$

Example Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$ for $\theta \in \mathbb{R}$.



[Solution: Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1-\cos\theta)d\theta$, we have $A = \int_0^{2\pi r} y\,dx = r^2 \int_0^{2\pi}(1-\cos\theta)^2\,d\theta = 3\pi r^2$.]

Arc Length

Recall that if the curve C is defined by $y = F(x)$, for $a \leq x \leq b$, and if $F'(x)$ is continuous on $[a,b]$, then the length L of C is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Suppose that

$$\begin{aligned} C &= \{(x,y) \mid y = F(x),\ x \in [a,b]\} \\ &= \{(x,y) \mid x = f(t),\ y = g(t),\ \ f : [\alpha,\beta] \to [a,b] \text{ is } 1-1, \text{ onto with } \frac{dx}{dt} = f'(t) > 0 \text{ for } t \in [\alpha,\beta]\} \end{aligned}$$

i.e. C is traversed once, from $x = a$ to $x = b$ as t increases from α to β and $f(\alpha) = a,\ f(\beta) = b$

Then we can calculate a length by using the substitution rule as follows:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Theorem Suppose that

- C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$,
- $\dfrac{dx}{dt} = f'(t)$ and $\dfrac{dy}{dt} = g'(t)$ are continuous for $t \in [\alpha, \beta]$,
- C is traversed exactly once as t increases from α to β.

Then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta ds, \quad \text{where } ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ for $\theta \in \mathbb{R}$.

[Solution: $L = r\displaystyle\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = r\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = 2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right)\, d\theta = 8r$.]

Definition

- Let f, g be continuously differentiable functions defined on I,
- let C be a parametric curve defined by $x = f(t)$, $y = g(t)$ for $t \in I$
- let $s(t)$ be the arc length along C from an initial point $(f(\alpha), g(\alpha))$ to a point $(f(t), g(t))$ on C.

Then the arc length function s for parametric curves is defined by

$$s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du.$$

Note that if parametric equations describe the position of a moving particle (with t representing time), then the **speed** of the particle at time t, is

$$v(t) = s'(t) = \frac{d}{dt}\int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

by the Fundamental Theorem of Calculus.

Surface Area

Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$, $g(t) \ge 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about the x-axis, then the area $A(S)$ of the resulting surface S is given by

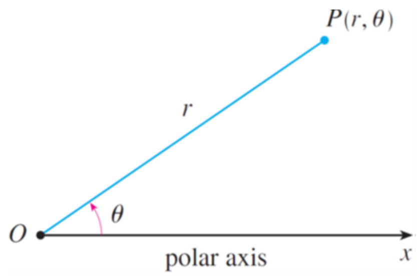
$$A(S) = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi y\, ds.$$

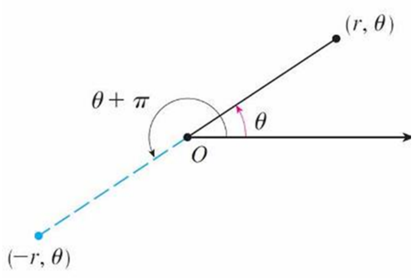
Example Show that the surface area of a sphere of radius r is $4\pi r^2$.

[Solution: The sphere S is obtained by rotating the semicircle $x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$, about the x-axis.Therefore, we get $A(S) = \int_0^\pi 2\pi r \sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = 4\pi r^2$.]

Polar Coordinates

In two dimensions, the Cartesian coordinates (x, y) specify the location of a point P in the plane. Another two-dimensional coordinate system is polar coordinates. Instead of using the signed distances along the two coordinate axes, polar coordinates (r, θ) specifies the location of a point P in the plane by its distance r from the origin O and the angle θ made between the positive x-axis and the line segment OP.





Remarks

- We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.
- Note that if $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the origin (or pole) for any value of θ.
- We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance r from O, but on opposite sides of O.

  Note that if $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the origin. So, $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$. In fact, (r, θ), $(r, \theta + 2n\pi)$ or $(-r, \theta + (2n + 1)\pi)$ represents the same point for any integer n.
- Use the following to find Cartesian coordinates $P(x, y)$ when polar coordinates $P(r, \theta)$ are given

  $$x = r\cos\theta, \qquad y = r\sin\theta$$

  and use the following to find polar coordinates $P(r, \theta)$ when Cartesian coordinates $P(x, y)$ are given

  $$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x} \quad \text{provided that } x \neq 0.$$

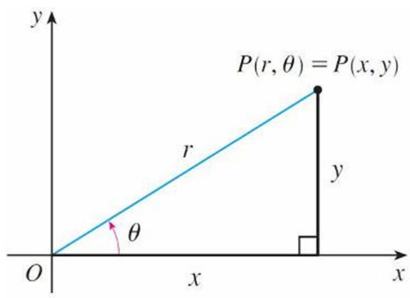
  Note that if $x \neq 0$, since $-\frac{\pi}{2} < \theta = \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{2}$, and $\tan\theta$ is a periodic function with period equals to π, i.e. $\tan(\theta + \pi) = \tan\theta$, we must choose between θ and $\theta + \pi$ so that the point (r, θ) lies in the correct quadrant.

Examples

1. Plot the points whose polar coordinates are given.

   (a) $(1, \frac{5\pi}{4})$ (b) $(2, 3\pi)$ (c) $(2, \frac{-2\pi}{3})$ (d) $(-3, \frac{3\pi}{4})$

2. Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates. [Solution: $(1, \sqrt{3})$.]

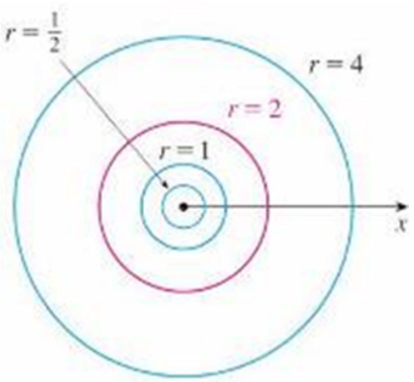


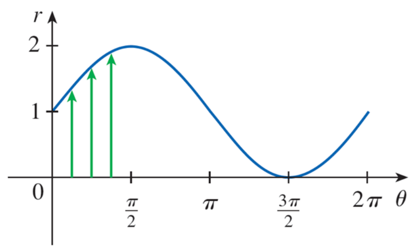
3. Represent the point with Cartesian coordinates $(1,-1)$ in terms of polar coordinates.
   [Solution: $(\sqrt{2}, \frac{-\pi}{4})$ or $(\sqrt{2}, \frac{7\pi}{4})$]

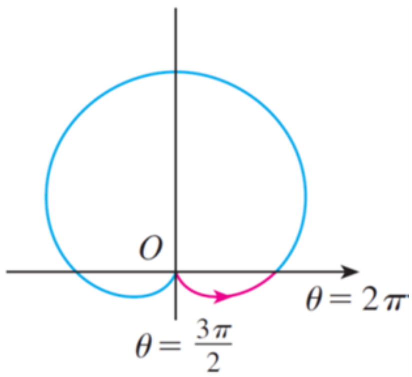
Definition The graph of a polar equation $r = f(\theta)$, or more generally $F(r,\theta) = 0$, is a set of all points $P(r,\theta)$ satisfying the equation. For example, the graph of $r = f(\theta)$, $\theta \in$ domain of f is the set $\{(r,\theta) \mid r = f(\theta),\ \theta \in \text{domain of } f\}$.

Examples

1. What curve is represented by the polar equation $r = 2$?
   [Solution: A circle with center O and radius 2.]
2. Sketch the curve $r = 1 + \sin\theta$. [Solution: A cardioid. Note that $\sin(\pi - \theta) = \sin\theta$, the cardioid $r = 1 + \sin\theta$ is symmetric about y-axis.]







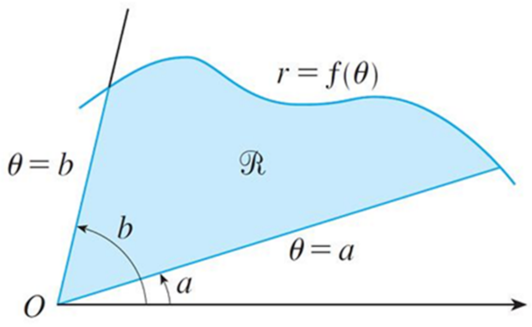
3. Sketch (a) the ray (or line) $\theta = 1$, (b) the circle $r = 2\cos\theta$ and (c) the four-leaved rose $r = \cos 2\theta$. [Note that $\cos(-2\theta) = \cos(2\theta)$, the four-leaved rose $r = \cos 2\theta$ is symmetric about x-axis.]

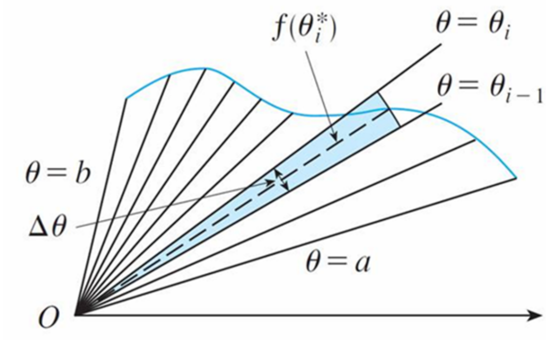
Calculus in Polar Coordinates

Area To develop the formula for the area of a region whose boundary is given by a polar equation, we need to use the formula for the area of a sector of a circle $A = \frac{1}{2}r\theta$, where r is the radius and θ is the radian measure of the central angle.

Let $\mathscr{R}$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width





$\Delta\theta$. Then the total area A of $\mathscr{R}$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i^*)]^2 \, \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 \, d\theta.$$

Example Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

[Solution: $A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta = \int_0^{\pi/4} \cos^2 2\theta \, d\theta == \int_0^{\pi/4} \frac{1+\cos 4\theta}{2} \, d\theta = \frac{\pi}{8}$.]

Arc Length and Tangents Let $f(\theta)$ be a continuously differentiable function for $\theta \in [a, b]$ and let C be the polar curve defined by $r = f(\theta)$, $a \leq \theta \leq b$. Since

$$\begin{aligned} & x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta \\ \Longrightarrow \quad & \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta, \end{aligned}$$

and using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2. \end{aligned}$$

Thus the length L of a curve C with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

Examples

1. Find the length of the cardioid $r = 1 + \sin\theta$.

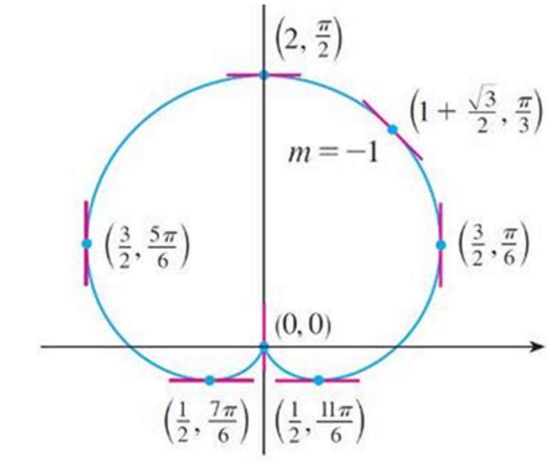
   [Solution: Since $r = 1 + \sin\theta$ is symmetric about y-axis, $L = \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{1-\sin\theta}} \, d\theta - \int_{\pi/2}^{3\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{1-\sin\theta}} \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{2\sqrt{2}\cos\theta}{\sqrt{1-\sin\theta}} \, d\theta = -4\sqrt{2}\sqrt{1-\sin\theta}\big|_{-\pi/2}^{\pi/2} = 8$.]

2. For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

   [Solution: $\frac{dy}{dx}|_{\theta=\pi/3} = \frac{dy/d\theta}{dx/d\theta}|_{\theta=\pi/3} = -1.$]

3. Find the points on the cardioid where the tangent line is horizontal or vertical.

   [Solution: Since $y = r\sin\theta = \sin\theta + \sin^2\theta$, $x = r\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta$, $\frac{dy}{d\theta} = \cos\theta + \sin 2\theta = \cos\theta(1 + 2\sin\theta) = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$, and $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = (1+\sin\theta)(1-2\sin\theta) = 0$ when $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$, there are horizontal tangents at the points $(r,\theta) = (2, \frac{\pi}{2}), (\frac{1}{2}, \frac{7\pi}{6}), (\frac{1}{2}, \frac{11\pi}{6})$ and vertical tangents at $(r,\theta) = (\frac{3}{2}, \frac{\pi}{6})$ and $(\frac{3}{2}, \frac{5\pi}{6})$.]



4. Sketch the graph of $r = 3\cos(3\theta)$.

   Since $r = 3\cos(3\theta) = 3\cos(-3\theta)$, the graph is symmetric about x-axis, it suffices to sketch the graph of $r = 3\cos(3\theta)$ for $\theta \in [0, \pi]$. Also note that if $r > 0$, then $(-r, \theta) = (r, \theta + \pi)$. Since

	$(0, \pi/6)$	$(\pi/6, \pi/2)$	$(\pi/2, 5\pi/6)$	$(5\pi/6, \pi)$
$r = 3\cos(3\theta)$	> 0	< 0	> 0	< 0

   the graph of $r = 3\cos(3\theta)$ is drawn in blue color for $\theta \in (\pi/6, \pi/2) \cup (5\pi/6, \pi)$.